Announcements

1) Example in 10/8
notes calculated

Recall: $v \in \mathbb{R}^3$,

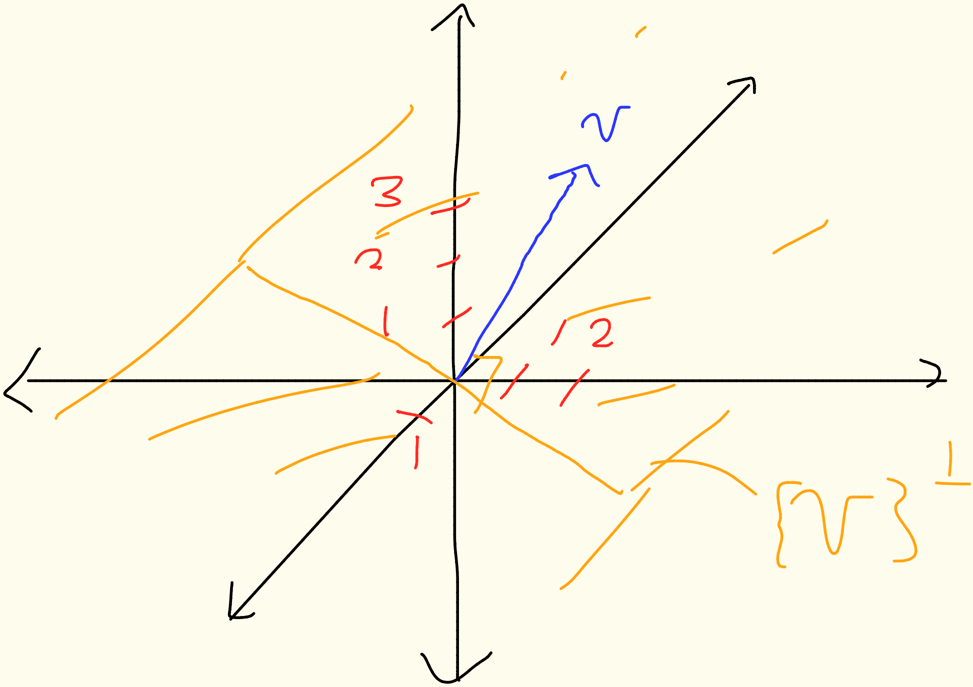
$v = (1,2,3)$. We

calculated $\{v\}^\perp$

and determined it

was a plane.

Picture



Proposition: ($S^\perp$, subspace)

Let S be a nonempty subset of an inner product space V.

Then $S^\perp$ is a subspace of V.

Proof: Recall that if $\langle \cdot, \cdot \rangle$ denotes the inner product on V,

$$S^{\perp} = \{ w \in V \mid \langle u, w \rangle = 0 \ \forall u \in S \}.$$

It is then immediate that $0_V \in S^{\perp}$ since $\langle x, 0_V \rangle = 0$ $\forall x \in V$.

We use the subspace test.

If $w, x \in S^\perp$ and $u \in S$,

then $\langle u, w \rangle = \langle u, x \rangle = 0$.

Therefore,

$$\langle u, w - x \rangle$$

$$= \langle u, w \rangle - \langle u, x \rangle$$

$$= 0 - 0 = 0$$

If α is any scalar, then if $w \in S^\perp$ and $u \in S$,

$$\langle u, \alpha w \rangle = \bar{\alpha} \langle u, w \rangle$$

$$= \bar{\alpha} \cdot 0$$

$$= 0$$

By the subspace test, $S^\perp$ is a subspace. $\square$

Theorem: (orthogonal decomposition)

Let V be an inner product space and let $W \subseteq V$ be a finite-dimensional subspace of V. Then $\forall\, x \in V$, $\exists$ unique vectors $P_x \in W$ and $P_x^{\perp} \in W^{\perp}$ with $\boxed{x = P_x + P_x^{\perp}}$

Proof: Suppose $\dim(W) = n < \infty$.

Let $\{w_1, w_2, \dots, w_n\}$ be a basis for W. Apply Gram-Schmidt to obtain an orthogonal basis $\{x_1, x_2, \dots, x_n\}$ for W.

Write

$$Px = \sum_{i=1}^{n} \langle x, x_i \rangle x_i$$

$$\in W.$$

Set $P^{\perp}x = x - Px$.

Let $y \in W$. Show

$\langle P^{\perp}x, y \rangle = 0$.

Write $y = \sum_{i=1}^{n} \alpha_i x_i$.

Then

$$\langle P^{\perp}x, y\rangle$$

$$= \langle x - Px, y\rangle$$

$$= \langle x, y\rangle - \langle Px, y\rangle$$

$$= \langle x, y\rangle - \left\langle \sum_{j=1}^{n} \langle x, x_j\rangle x_j, y\right\rangle$$

$$= \langle x, y\rangle - \left\langle \sum_{j=1}^{n} \langle x, x_j\rangle x_j, \sum_{i=1}^{n} \alpha_i x_i\right\rangle$$

$$= \langle x, y \rangle - \left\langle \sum_{j=1}^{n} \langle x, x_j \rangle x_j , \sum_{i=1}^{n} \alpha_i x_i \right\rangle$$

$$= \langle x, y \rangle - \sum_{j=1}^{n} \sum_{i=1}^{n} \left\langle \langle x, x_j \rangle x_j , \alpha_i x_i \right\rangle$$

$$= \langle x, y \rangle - \sum_{j=1}^{n} \sum_{i=1}^{n} \langle x, x_j \rangle \bar{\alpha}_i \underbrace{\langle x_j , x_i \rangle}_{0 \text{ if } i \neq j}$$

$$= \langle x, y \rangle - \sum_{j=1}^{n} \langle x, x_j \rangle \bar{\alpha}_i$$

$$= \langle x, y \rangle - \langle x, y \rangle = 0 .$$

We now know

$Px \in W$, $P^{\perp}x \in W^{\perp}$.

Now suppose we can

write $x = y + z$ where

$y \in W$, $z \in W^{\perp}$. We

want to show $y = Px$,

$z = P^{\perp}x$.

Then

$$0_V = x - x$$
$$= (p_x + p_x^\perp) - (y + z).$$

So

$$0 = \langle x - x, x - x \rangle$$

$$= \langle (p_x + p_x^\perp) - (y+z), (p_x + p_x^\perp) - (y+z) \rangle$$

$$= \langle \underbrace{(p_x - y)}_{\in W} + \underset{\in W^\perp}{(p_x^\perp - z)}, (p_x - y) + (p_x^\perp - z) \rangle$$

$$= \langle (Px-y),(Px-y)\rangle$$

$$+ \langle (\underbrace{Px-y}_{\in W}), (\underbrace{P^{\perp}x-z}_{\in W^{\perp}})\rangle$$

$$+ \langle (\underbrace{P^{\perp}x-z}_{\in W^{\perp}}), (\underbrace{Px-y}_{\in W})\rangle$$

$$+ \langle (P^{\perp}x-z),(P^{\perp}x-z)\rangle$$

$$= \langle (Px-y),(Px-y)\rangle$$

$$+ \langle (P^{\perp}x-z),(P^{\perp}x-z)\rangle$$

Since $\langle (Px-y),(P^{\perp}x-z)\rangle = 0$.

Since this last quantity is

$$\| Px - y \|_2^2 + \| P^{\perp} x - z \|_2^2$$

which is then equal to zero by our initial assumption,

$$Px - y = 0_V = P^{\perp} x - z$$

$$\Rightarrow \quad y = Px, \; z = P^{\perp} x.$$

$\square$

Remark: It is an indirect consequence of the proof that

$$\|x\|_2^2 = \|Px\|_2^2 + \|P^\perp x\|_2^2 .$$

Theorem: (nearest element)

If V is an inner product space and W is a finite-dimensional subspace of V, then if $x \in V$, the vector $Px \in W$ minimizes the 2-norm distance from x to W,

i.e. $\|x - Px\|_2 = \min_{y \in W} \|x - y\|_2$

Proof: Take $y \in W$.

Then

$$\|x-y\|_2^2 = \|Px + P^\perp x - y\|_2^2$$

$$= \|\underbrace{(Px-y)}_{\in W} + \underbrace{P^\perp x}_{\in W^\perp}\|_2^2$$

$$= \|(Px-y)\|_2^2 + \|P^\perp x\|_2^2$$

which is minimized when

$\|(Px-y)\|_2 = 0$, i.e. $Px = y$. $\square$

Definition: (orthogonal projection)

If V is an inner product space and $W \subseteq V$ is a finite-dimensional subspace,

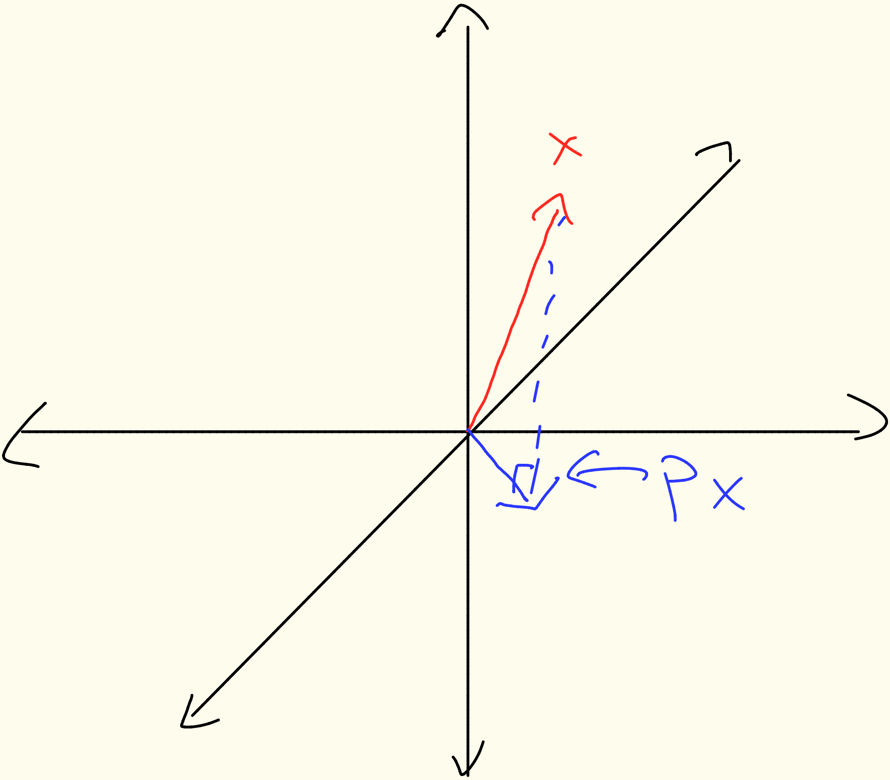
then the map

$P: V \to W$ given by

$x \mapsto Px$ (Px is the vector in W defined earlier) is

the orthogonal projection of V onto W.

Picture



$$W = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$$

Px is the "shadow" of x on W.